

## INTERVAL ESTIMATION USING REPLICATION/DELETION AND MSER TRUNCATION

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### ABSTRACT

This paper addresses the construction of a consistent interval estimator for the steady-state mean within a replication/deletion framework for output analysis when MSER truncation is applied. Because the MSER truncation point is a random variable, the truncated output sequences for each replication typically are unequal in length. A weighting scheme is applied to the replication means to correct for unequal sample sizes, as is standard in ANOVA. A numerical example is provided to illustrate the procedure and consequences.

### 1 INTRODUCTION

Over a half-century ago, Conway recognized the initial transient as the first among three principle tactical problems in steady-state simulation (Conway 1959; Conway, *et al.* 1963; Goldsman, *et al.* 2010). An arbitrary selection of initial conditions for simulation runs introduces bias in the estimation of output statistics, such as the stationary mean. The most common approach for mitigating such bias is to truncate or delete some number of observations from the beginning of the output sequence and to compute statistics using only the remaining observations.

Alternate criteria for determining a good truncation point—one that adequately removes the bias without undue loss of precision—have been the subject of continuing invention beginning with Conway himself. Recently, however, a consensus has emerged among researchers that MSER has all of the properties most desired in a truncation criterion. It is effective and efficient at mitigating bias, robust across alternate forms of biasing functions, computationally trivial, easily understood, and does not require experimenter intervention to establish parameters.

MSER was initially developed by McClarnon (1990), White and Minnox (1994), and White (1997) and was applied and extended by Rossetti *et al.* (1995), Spratt (1998), Cobb (2000), White, *et al.* (2000), and Franklin (2009). Mahajan and Ingalls (2004) determined three truncation criteria adequate, with MSER-5 recommended for its efficiency and robustness. Oh and Park (2006) compared their EVR method “with the method MSER-m known as the most sensitive rule in detecting bias and most consistent rule in mitigating its effects.” MSER was shown to outperform EVR in almost all experiments. Sandikci and Sabuncuoglu (2006) automated MSER-5 as their means for studying transients. Bertoli, Casale, and Serazzi (2007, 2009) selected MSER-5 as the initialization approach for their Java Modeling Tools package and included a usage wizard. The criterion gained additional traction with an exhaustive empirical evaluation by Hoad, *et al.* (2008, 2011), who chose MSER-5 as the most suitable for automation over a wide range of published approaches to the transient problem, including heuristics, graphical procedures, initialization bias tests, statistical methods, and hybrid approaches.

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White and Franklin (2010) confirm the empirical findings of White and Robinson (2010) regarding the relationship between the MSER truncation point and the degree of mean bias and autocorrelation in an output sequence. They introduce a parametric approach to analyzing the expected behavior of MSER and apply this approach to an output model with geometrically decaying bias and constant-parameter AR(1) white noise. Franklin *et al* (2009) explore the intuition that MSER minimizes the mean squared error (MSE) of the mean estimator. This empirical result is confirmed by Pasupathy and Schmeiser (2010). They reason that MSE is the most appropriate criterion for evaluating alternate truncation criteria, show that the MSER statistic is asymptotically proportional to the MSE, and conclude that the MSER statistic is a solid foundation for initial-transient algorithms. Pasupathy and Schmeiser also suggest two new algorithms using the MSER statistic and compare these to the original MSER algorithm using empirical results for M/M/1 and AR(1) data processes. Mokashi *et al* (2010) compared their N-Skart method with MSER-5 and achieved only modest improvements with considerably greater computational effort. Most recently, Hoad and Robinson (2011) consider the practical implementation of MSER-5.

## 2 TRUNCATION AND THE MSER CRITERION

Denote the output of a single replication of a simulation as the time series  $[y_i: i=1, 2, \dots, n]$ . Truncation divides this into two subseries  $[(y_i: i=1, 2, \dots, d), (y_i: i=d+1, 2, \dots, n)]$ , where  $d$  is the truncation point. For an output that is tallied, under truncation the estimator for the mean output is the sample mean of the second (reserved) subseries

$$\bar{Y}(n, d | y_0) = \frac{1}{n-d} \sum_{i=d+1}^n Y_i$$

The MSER criterion for the optimal truncation point is

$$d^* = \arg \min_{n \gg d \geq 0} [MSER(n, d | y_0)]$$

where the MSER statistic is the square of the estimated standard error of the mean  $MSER(n, d | y_0) = \tilde{SE}_{\bar{Y}}^2(n, d | y_0) = S_{\bar{Y}}^2(n, d | y_0) / n$  obtained using the large-sample variance

$$S_{\bar{Y}}^2(n, d | y_0) = \frac{1}{(n-d)^2} \sum_{i=d+1}^n (Y_i - \bar{Y}(n, d | y_0))^2$$

$n \gg d \geq 0$

Note that for correlated data the sample variance is a biased estimator. For a covariance-stationary process the *actual* squared standard error is

$$SE_{\bar{Y}}^2(\bar{x}) = \frac{\sigma^2}{n} \left[ 1 + 2 \sum_{i=1}^{n-1} \left( 1 - \frac{i}{n} \right) \rho_i \right]$$

where  $\sigma^2$  is the lag-zero autocovariance and  $\rho_k$  is the lag  $k$  –lag autocorrelation. Stationarity requires that the bracketed term is finite as  $n \rightarrow \infty$ . In practical terms this means that for sufficiently large  $n$  the bracketed term becomes *de facto* a constant. Therefore we can consider  $SE_{\bar{Y}}^2 = c\sigma^2/n$  and estimate it as  $\bar{SE}_{\bar{Y}}^2 = cS_{\bar{Y}}^2/n$ . Since the MSER truncation point is based on *minimizing*  $SE_{\bar{Y}}^2$  (rather than *estimating* it),

we can effectively ignore the constant and determine a suitable MSER truncation point based on just the familiar variance estimator. This explains why Franklin and White (2008) found that, as expected, the Phillips-Perron variance estimator performed no better in practice than the naïve variance estimator.

### 3 APPLYING MSER WITHIN A REPLICATION/DELETION FRAMEWORK

Denote the output of  $m$  independent replications of a simulation as the set of  $m$  time series  $[y_{ij}; i=1,2,\dots,n_i; j=1,\dots,m]$ . Without loss of generality, consider that each of these series has the same initial condition  $y_0$  and the same run length  $n_j=n \forall j$ . As before, truncation divides each time series into two sub-series  $[(y_{ij}; i=1,2,\dots,d_j), (y_i; i=d_j+1,2,\dots,n; j=1,\dots,m)]$ , where  $d_j$  is the truncation point for the  $j^{\text{th}}$  series. For an output that is tallied, under replication/deletion we obtain a random sample of  $m$  values for the mean, each estimated from one of the reserved subseries as the corresponding sample mean

$$\bar{Y}_j(n, d_j | y_0) = \frac{1}{n - d_j} \sum_{i=d_j+1}^n Y_{ij}; j = 1, \dots, m$$

Note, however, that the MSER truncation point for the  $j^{\text{th}}$  replication is the integer random variable  $D_j^*$ . This means that attempts to create interval estimators using run-based replication (Conway's second principle problem) are biased if constructed from independent point estimates based on different sample sizes. Two solutions present themselves immediately: (1) find and apply the maximum truncation amount to all runs to reduce them to a common size; or (2) use weighted estimators for both the mean and variance. In this paper we will investigate the second option, which is standard in ANOVA and preserves as much usable data as possible.

Denote the total number of observations reserved across all runs as  $N = \sum_{j=1}^m (n - d_j^*)$ . We will regard the  $\bar{Y}_j$ 's as having a common expected value and underlying variance, but different standard errors because of the different sample sizes. We adjust for this using a weighted average

$$\bar{\bar{Y}} = \sum_{j=1}^m w_j \bar{Y}_j$$

If  $\sum_{j=1}^m w_j = 1$ , i.e., we have a convex combination, then  $\bar{\bar{Y}}$  is an unbiased estimator of the common mean. A well-known result is that variance of  $\bar{\bar{Y}}$  is minimized when  $w_j = (n - d_j^*)/N$ . Finally, in order to obtain an interval estimator we need an estimator for the variance of  $\bar{\bar{Y}}$

$$Var(\bar{\bar{Y}}) = \frac{1}{1 - \sum_{j=1}^m w_j^2} \sum_{j=1}^m w_j (\bar{Y}_j - \bar{\bar{Y}})^2$$

which is an unbiased estimator with  $m(1 - \sum_{j=1}^m w_j^2)$  degrees of freedom. Note that if the weights are all equal ( $w_j = 1/m \forall j$ ) this reduces to the familiar sample variance formula with  $m-1$  degrees of freedom. Note also that for unequal weighting the degrees of freedom in general will not be integer and the corresponding  $t$ -value will need to be generated with software.

#### 4 AN EXAMPLE

We ran a simulation of an M/M/1 queueing system at traffic intensity 0.95 (arrival rate = 19/time unit, service rate = 20/time unit) for  $n=10,000$  and  $n=100,000$  observations. We used delay in queue as our performance measure, which is suitable for tally statistics as described in Section 3. The output from each of 10 runs was truncated based on MSER-5 and the 10 resulting sample means were pooled using the weighting process described above to form a 90% confidence interval. This process was repeated 1,000 times to create 1,000 confidence intervals. Nominally we would expect 90% of such intervals to cover the true answer ( $\mu = 0.95$ ) obtained from theory. We created an indicator variable for each confidence interval to record whether it covered or did not cover the true mean. The results were analyzed using JMP-9 and are summarized in Figure 1.

Using a run length of 10,000 we obtained empirical coverage of 79% for a nominal 90% CI. When run lengths of 100,000 were used, the empirical coverage improved to 86.5%. Since all CIs were based on 10 runs, the degrees of freedom would be 9 if all run lengths were equal. Actual degrees of freedom varied, ranging from 9 down to 8.68 in our 2000 sets of experiments. Using a conservative 8 degrees of freedom had virtually no impact on the coverage, improving it only from 86.5% to 86.9% when run lengths of 100,000 were used. Upon inspection, failure to achieve nominal coverage seems to be because the confidence intervals are centered at the MSER estimate of the mean, which is known to be biased. As we observed, the bias has a greater impact when shorter runs are used.



Figure 1: Statistics of the indicator variable recording coverage of the 90% confidence interval on the mean delay in queue for a simulated M/M/1 queue.

## 5 CONCLUSION

In this paper we provide a brief overview of the MSER truncation criterion, a technical note on the construction of interval estimates using replication/deletion and MSER truncation, a weighting scheme that can be applied to this end, and an example of such an application. Empirical results from the example suggest that the weighting of unequal samples modestly underestimates the width of confidence intervals on the mean, with underestimation decreasing as a function of increasing run lengths.

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